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Foundations of Time Evolution in Quantum Mechanics

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Hypotheses and postulates for time evolution are formulated in an algebraic setting.

This paper is an invitation to dig deeper into the foundations of time evolution, to find its fundamental proofs in the deeper logicoalgebraic structures which carry the Hilbert space formalism. A derivation of the Schrödinger equation inside the Hilbert space formalism can be found in Piron (1976). There, one has to consider the distinction between mixed and pure states, and one has to deal with the question of whether pure states remain pure. These problems are deeply connected with the interpretation of the quantum mechanical formalism; so I think it is a considerable relief to discover the possibility of not having to discuss them. It turns out that the invariance of mixing properties under time evolution is a consequence of very general postulates.

Before going into the details, it is appropriate to define the meaning of "time" for a physicist.

TIME HAS SEVERAL APPEARANCES IN PHYSICS

A1: In general relativity it is itself an *object* of the theory. Unified with space, it is in mutual interaction with matter.

A2: For smaller systems, it is an external parameter, used to describe the *dynamics*, the *time evolution.*

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A3: There is a *symmetry* of all the physical laws; they are invariant under *translation* in time and, more generally, under transformations belonging to the Poincaré or Galilei group.

A4: The basic laws are also invariant under *time reflection*, if coupled with changes of charges and of parity.

A5: For large systems, there is the *arrow of time*, a breaking of time-reversal symmetry, stated as the *second law of thermodynamics.*

A6: There is an *arrow of time* in *cosmology.*

A7: There is also an *arrow of time* in the *measurement* process. The "collapse of the wave packet" is irreversible.

For the problem of the evolution of small systems we work in the setting of A2. Next we state analogs of Newton's hypotheses of absolute time and absolute space including the definition of a "time evolution," followed by the postulates describing its properties.

HYPOTHESES

H1: Time is an external parameter, equipped with a pair relation between different points in time: "t is later than s," abbreviated as $t > s$.

H2: The set of observables at a fixed time t forms a C^*-algebra $\mathcal{A}_t$, which has an irreducible representation in a Hilbert space $\mathcal{H}$. This representation is unique up to unitary equivalence. The possible physical states of the system are represented by the mathematical states of the algebra $\mathcal{S}(\mathcal{A}_t)$.

H3: The Hilbert space $\mathcal{H}$ is separable (it has a countable basis).

H4: The set of observables is at each time the same:

$$\forall s, \forall t: \quad \mathcal{A}_s = \mathcal{A}_t =: \mathcal{A}$$

H5: For the time t later than s, there corresponds to each state $w_s \in \mathcal{S}(\mathcal{A}_s)$ one and only one state $w_t = T_{t,s}w_s \in \mathcal{S}(\mathcal{A}_t)$. The mapping $T_{t,s}$: $\mathcal{S}(\mathcal{A}_s) \to \mathcal{S}(\mathcal{A}_t)$ represents the **time evolution.**

POSTULATES

P1: *Affinity (superposition principle):* The time evolution is compatible with the mixing of states. $\forall\{v_s, w_s\} \subset \mathcal{A}_s$, $\forall\lambda \in [0, 1]$:

$$T_{t,s}(\lambda v_s + (1 - \lambda)w_s) = \lambda T_{t,s}v_s + (1 - \lambda)T_{t,s}w_s$$

P2: *Partial reversibility:* There exists not only a time evolution to later times, into the "future," but there is also a law to find the "past"; $T_{t,s}$ is *injective*:

$$\forall t,\, s,\, t > s\colon \exists T_{t,s}^{-1}\colon \quad \{w_t = T_{t,s}w_s \,|\, w_s \in \mathcal{A}_s\} \to \{w_s\} = \mathcal{S}(\mathcal{A}_s)$$

$$T_{t,s}^{-1}T_{t,s} = \text{identity}\,|_{\mathcal{S}(\mathcal{A}_s)}$$

P3: *Surjectivity:* Each state can be reached by the time evolution; for t later than s, each state at time t does have a past history at time s; $T_{t,s}$ is *surjective*:

$$\{T_{t,s}w_s \,|\, w_s \in \mathcal{S}(\mathcal{A}_s)\} = \mathcal{S}(\mathcal{A}_t)$$

P4: *Divisibility:* Time evolution can be split over times in between:

$$u > t > s \Rightarrow T_{u,s} = T_{u,t}T_{t,s}$$

P5: *Invariance under time translation:* The time differences $\{\tau =: t - s \,|\, t > s\}$ form a semigroup under addition, and the time evolution is determined by time differences only:

$$\forall s,\, \forall t\colon \quad T_{s+\tau,s} = T_{t+\tau,t} =: T_\tau$$

P6: *Measurability:* The time differences form an interval of real numbers. The expectation values form measurable functions:

$$\forall \tau,\, \forall a \in \mathcal{A},\, \forall w \in \mathcal{S}(\mathcal{A})\colon \quad \tau \mapsto (T_\tau w)(a) \quad \text{is measurable}$$

In the Heisenberg picture one has another conception of time evolution and we have to state other postulates.

THE HEISENBERG SCHEME

$H5_H$: For the time t later than s, there corresponds to each observable $a_t \in \mathcal{A}_t$ one and only one observable $a_s = \mathcal{S}_{s,t}a_t \in \mathcal{A}_s$. The mapping $S_{s,t}\colon \mathcal{A}_t \to \mathcal{A}_s$ represents the time evolution. It is linear:

$$S_{s,t}(a + \alpha b) = S_{s,t}a + \alpha S_{s,t}b$$

it maps positive observables to positive observables and the unity to the unity.

$P2_H$: *Surjectivity:* Each element $a_s \in \mathcal{A}_s$ is the image of some $a_t \in \mathcal{A}_t$ under the mapping $S_{s,t}$.

$P3_H$: *Homomorphicity:* The mapping $S_{s,t}$ is an algebraic *-homomorphism:

$$S_{s,t}(ab) = (S_{s,t}a)(S_{s,t}b)$$

$$S_{s,t}(a^*) = (S_{s,t}a)^*$$

P4$_H$: *Divisibility:* $u > t > s \Rightarrow S_{s,u} = S_{s,t}S_{t,u}$.

P5$_H$: *Invariance under time translation:* The time differences form a semigroup under addition and

$$\forall s, \forall t, \forall \tau: \quad S_{s,s+\tau} = S_{t,t+\tau} =: S_\tau$$

P6$_H$: *Measurability:* The time differences form an interval of real numbers, and

$$\forall \tau, \forall a \in \mathcal{A}, \forall w \in \mathcal{S}(\mathcal{A}): \quad \tau \mapsto w(S_\tau a) \quad \text{is a measurable function}$$

In Baumgartner (1992, 1994) one can find the proofs of necessity and sufficiency, and moreover, some philosophical remarks on the postulates, with conjectures on their deeper roots, connecting A2 with other appearances of time.

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